# A new E-field finite element formulation for the numerical modelling of high temperature superconductors


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***Abstract*—In this work we will present a new 3-D full-wave finite element formulation and its application to the numerical modelling of high temperature superconductors. This new approach has the electric field as the main unknown, which allows the direct calculation of the induced eddy currents and other magnitudes of interest (e.g. Joule heating) without resorting to derivatives. It does not require the addition of extra unknowns (e.g Lagrange multipliers or scalar potentials) to be stable and it can account for capacitive and inductive effects simultaneously, even at low frequencies. The new approach applies mass-lumping $L^2$ projections on the curl and divergence operators of the regularized Maxwell equations weak form and uses first order Lagrangian nodal finite elements enriched with an inner bubble. In this work we will also explore the possibility of converting the time-dependent non-linear problem in several parallel non-linear problems in frequency domain, which can be solved independently (avoiding sequential time-stepping) with the consequent savings of computational costs.**



***Keywords*—*Finite element method, regularized Maxwell equations, $L^2$ projection, nodal elements, bubble elements.***


## I. Introduction

The success of new compact nuclear fusion reactors depends in great measure on the capacity to generate large magnetic fields to confine the plasma. The design of coils capable to produce these fields will greatly benefit from the development of numerical models and tools able to predict their behaviour under the extreme conditions that they will suffer in a nuclear fusion environment. High temperature superconductors are a good candidate to carry the large currents required to create large fields but the numerical modelling of these materials is numerically challenging and demanding [1].

The objective of this work is to propose a numerical model able to provide relevant engineering data to facilitate design decisions. We have selected the electric field as the main variable because the currents and heating losses are usually the magnitude of interest [2]. Also, we want to consider capacitive effects (e.g. arcing, for nuclear safety certification purposes) and the influence that dielectric insulators can have on the performance of the coils. And, of course, we want to make the tool as fast as possible to simulate the maximum amount of design possibilities in the shortest possible time. The numerical approach we propose is explained in the next section and it delivers all the requirements we are looking for.

## II. Finite Element Formulation

The finite element approach proposed in this work is based on the regularized formulation described in [3] and implemented in the open-source code ERMES [4]. The regularized formulation [3] meets all the requirements stated at the beginning of this article, i.e., (1) it calculates the electric field without resorting to derivatives, (2) no extra unknowns are necessary for numerical stability, (3) the resultant matrix is well conditioned (i.e. it can be solved easily with a lightly preconditioned iterative solver [5]), and (4) it is able to calculate capacitive and inductive effects simultaneously at any frequency. However, as it is mentioned in [3] and [4], it is necessary to take special measures when dealing with field singularities and discontinuities, i.e. we must remove the divergence term on the elements around a field singularity and use a double-node technique on the surfaces of discontinuity between different media. Moreover, in the presence of singularities, we must use second order (or higher) Lagrangian nodal elements to make the formulation converge to the physical solution.

The application of these special measures requires some extra effort by the ERMES user. Then, to make things simpler, we have modified [3] and [4] in such a way that is not longer necessary to use these measures and, at the same

time, we keep the beneficial properties of the formulation [3]. This modification is described in [6] and it consists of the application of mass-lumping $L^2$ projections on the curl and divergence operator of the regularized formulation [3]. This technique is used in combination with first order Lagrangian nodal finite elements enriched with an inner bubble [6].

Before presenting the mathematical description of the proposed formulation, we need first to introduce the following functional spaces [6]:

$$\begin{aligned} B_h &= \{\mathbf{v} \in (H_0^1)^3 \,:\, \mathbf{v}|_K \in b_K(P_0(K))^3, K \in \Im_h\} \\ V_h &= \{q \in H^1 \,:\, q|_K \in P_1(K), K \in \Im_h\} \\ U_h &= (\,(V_h)^3 \cap H_0(curl)\,) + B_h \end{aligned} \tag{1}$$

where $B_h$ is the bubble function space, $V_h$ is a nodal-continuous linear finite element space and $U_h$ is the solution space. $\Im_h$ is a conforming discretization of the problem domain $\Omega$ into tetrahedrons $K$. $P_l(K)$ is the space of polynomials defined on $K$ of degree not greater than $l$. $b_K$ is the element bubble defined on $K$ by $b_K = N_1 N_2 N_3 N_4$ with $N_i$ being the nodal basis function associated with the vertex $i$ of $K$. $H^1$, $H_0^1$, $H(curl)$ and $H_0(curl)$ are the Hilbert spaces defined by:

$$\begin{aligned} H^1 &= \{q \in L^2(\Omega) \,:\, \partial q/\partial r \in L^2(\Omega)\,,\, r = x, y, z\} \\ H_0^1 &= \{q \in H^1 \,:\, q|_\Gamma = 0\} \\ H\,(curl) &= \{\mathbf{v} \in (L^2(\Omega))^3 \,:\, \nabla \times \mathbf{v} \in (L^2(\Omega))^3\} \\ H_0(curl) &= \{\mathbf{v} \in H(curl) \,:\, \hat{\mathbf{n}} \times \mathbf{v}|_\Gamma = 0\} \end{aligned} \tag{2}$$

with $L^2(\Omega)$ being the space of square integrable functions and $\Gamma$ a surface at the boundary $\partial\Omega$ of the domain $\Omega$.

The problem we solve to obtain the approximate electric field solution $\mathbf{E}_h$ consists in finding an $\mathbf{E}_h \in U_h$ such that $\forall\, \mathbf{V}_h \in U_h$ it is satisfied:

$$\begin{aligned} &(\, L_h(\,\mu^{-1}\nabla \times \mathbf{E}_h\,)\,,\, L_h(\,\mu^{-1}\nabla \times \mathbf{V}_h\,)\,)_{0,\mu,h} \\ &+\ (\,\hat{L}_h(\,\nabla \cdot \varepsilon \mathbf{E}_h\,)\,,\, \hat{L}_h(\,\nabla \cdot \bar{\varepsilon}\mathbf{V}_h\,)\,)_{0,\tau,h} \\ &-\ \omega^2(\,\varepsilon\mathbf{E}_h\,,\,\mathbf{V}_h\,)_\Omega \ -\ (\,\mu^{-1}\nabla \times \mathbf{E}_h\,,\,\hat{\mathbf{n}} \times \mathbf{V}_h\,)_{\partial\Omega} \\ &-\ (\,\tau^{-1}\nabla \cdot \mathbf{E}_h\,,\,\hat{\mathbf{n}} \cdot \mathbf{V}_h\,)_{\partial\Omega} \ =\ j\omega(\,\mathbf{J}\,,\,\mathbf{V}_h\,)_\Omega \end{aligned} \tag{3}$$

where $\mu$ is the magnetic permeability. $\varepsilon$ is the complex electrical permittivity ($\varepsilon = \epsilon + j\sigma/\omega$) and $\bar{\varepsilon}$ its complex conjugate. $\tau$ is the regularization term given in [3], which equal to $\varepsilon\bar{\varepsilon}\mu$. $\omega$ is the angular frequency and $j$ the imaginary unit. The $L^2$ inner product $(\cdot,\cdot)_\Omega$ is defined by:

$$(\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v} \tag{4}$$

and similarly for $\partial\Omega$. The mass-lumping discrete $L^2$ product $(\cdot,\cdot)_{0,\mu,h}$ is:

$$(\mathbf{u}, \mathbf{v})_{0,\mu,h} = \sum_{K \in \Im_h} \frac{|K|}{4} \sum_{i=1}^{4} \mu_i\ \mathbf{u}_i \cdot \mathbf{v}_i \tag{5}$$

being $|K|$ the volume of $K$ and $\mu_i$, $\mathbf{u}_i$, $\mathbf{v}_i$ the values of $\mu$, $\mathbf{u}$, $\mathbf{v}$ on the vertices of $K$. An equivalent definition is applied to the mass-lumping $L^2$ product $(\cdot,\cdot)_{0,\tau,h}$. Finally, the mass-lumping $L^2$ projections $L_h(\cdot)$ and $\hat{L}_h(\cdot)$ are defined for any given $\mathbf{u} \in (L^2(\Omega))^3$ by:

$$\begin{aligned} &(\, L_h(\nabla \times \mathbf{u})\,,\, \mathbf{v}\,)_{0,\mu,h} = (\,\mathbf{u}\,,\,\nabla \times \mathbf{v}\,) \quad \forall \mathbf{v} \in (V_h)^3 \\ &(\,\hat{L}_h(\,\nabla \cdot \varepsilon\mathbf{u})\,,\, q\,)_{0,\tau,h} = -(\,\mathbf{u}\,,\,\varepsilon\nabla q\,) \quad \forall q \in V_h \cap H_0^1. \end{aligned} \tag{6}$$

The non-linear relation of the electrical conductivity with the electric field $\sigma(\mathbf{E})$ is included in the imaginary part of $\varepsilon$. Problem (3) has been stated in frequency domain but, if a time domain formulation is preferred, a similar description can be developed by simply replacing the frequency terms by time derivatives ($-j\omega = \partial/\partial t$).

The reason to present (3) in frequency domain is because, after implementing (3) in ERMES, we are going to experiment with different possibilities of transforming the time-dependent non-linear problem of modelling a high temperature superconductor in a set of independent non-linear problems in frequency domain, which can be solved in parallel. We are going to adapt the developments described in [7]–[10] for non-linear magnetic materials to high temperature superconductors.

## III. Summary

We have presented a finite element formulation which is being implemented in the open-source code ERMES [4]. After finishing its implementation, our intention is to use this formulation to test the possibilities of transforming a time-dependant non-linear problem in a set of independent non-linear problems in frequency domain, which can be solved in parallel. This parallelisation will avoid the sequential waiting between time steps and it will improve in great measure the overall computational performance of the simulations.

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