Stability of H-A and T-A Finite Element Formulations

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***Abstract*—In this work, we present and analyze the numerical stability of two coupled finite element formulations. The first one is the h-a-formulation and is well suited for modelling systems with superconductors and ferromagnetic materials. The second one, the so-called t-a-formulation, applies for systems with thin superconducting domains. These formulations involve two coupled unknown fields and are mixed on the coupling interfaces. Function spaces in mixed formulations must satisfy compatibility conditions to ensure reliability and stability of the numerical solutions. We propose stable choices of function spaces using hierarchical basis functions and demonstrate the effectiveness of the approach on simple 2D examples.**

***Keywords*—Finite element analysis, high-temperature superconductors, mixed formulations, stability analysis.**

I. Introduction

In this work, we consider two coupled finite element formulations for modelling systems with high-temperature superconductors (HTS): the h-a-formulation for systems containing superconductors and ferromagnets [1] and the t-a-formulation for systems with thin superconducting tapes [2].

In these coupled formulations, different finite element fields are introduced region-wise, while they coexist and are coupled through a common boundary or a common region. The field coupling makes these formulations *mixed*, for which care must be taken in the choice of function spaces and the discretization. Naive choices of approximation function spaces can easily lead to stability issues manifesting themselves as spurious oscillations in the numerical solution [3].

Following the general theory of mixed finite elements, we analyze the related conditions for obtaining numerically stable mixed formulations. In particular, we perform the *inf-sup numerical test* for checking the compatibility of discretized function spaces. We restrict the analysis to 2D problems.

II. Finite Element Formulations

We model the magnetic response of HTS with the magnetodynamic equations, $\operatorname{div} \boldsymbol{b} = 0$, $\mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j}$, $\mathbf{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}$. In HTS, $\boldsymbol{b} = \mu_0 \boldsymbol{h}$ and we assume a power law for the resistivity ρ. The system can also contain ferromagnetic materials, that are assumed non-conducting, and for which the permeability is a function of $\boldsymbol{h}$.

The numerical domain Ω is decomposed into a conducting domain Ω_c, and its complementary domain $\Omega_\mathrm{c}^\mathrm{C}$. The notation $(A\,,B)_\Omega$ denotes the integral of the AB product on Ω.

A. Coupled formulation - h-a-formulation

Systems containing both HTS and nonlinear ferromagnetic materials are advantageously solved with a coupled h-a-formulation [1]. In this formulation, we decompose the domain Ω into two parts: Ω_h, containing the superconducting domain, to be solved with the h-formulation, and Ω_a, containing the nonlinear ferromagnetic domain, to be solved with the a-formulation. The common boundary of Ω_h and Ω_a is denoted as Γ_m. The coupled h-a-formulation writes:

From an initial solution, find $\boldsymbol{h} \in \mathcal{H}_{H,I}(\Omega_h)$ and $\boldsymbol{a} \in \mathcal{A}_A(\Omega_a)$ s.t., for $t > 0$, $\forall \boldsymbol{h}' \in \mathcal{H}_{0,0}(\Omega_h)$, and $\forall \boldsymbol{a}' \in \mathcal{A}_0(\Omega_a)$,

$$\begin{aligned} &\left(\partial_t(\mu\,\boldsymbol{h})\,,\boldsymbol{h}'\right)_{\Omega_h} + \left(\rho\,\mathbf{curl}\,\boldsymbol{h}\,,\mathbf{curl}\,\boldsymbol{h}'\right)_{\Omega_{h,\mathrm{c}}} \\ &\qquad + \left(\partial_t \boldsymbol{a} \times \boldsymbol{n}_{\Omega_h}\,,\boldsymbol{h}'\right)_{\Gamma_\mathrm{m}} = -\sum_{i\in C} V_i \mathcal{I}_i(\boldsymbol{h}'), \\ &\left(\boldsymbol{h} \times \boldsymbol{n}_{\Omega_a}\,,\boldsymbol{a}'\right)_{\Gamma_\mathrm{m}} - \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{a}\,,\mathbf{curl}\,\boldsymbol{a}'\right)_{\Omega_a} = 0, \end{aligned} \tag{1}$$

with $\mathcal{H}_{H,I}(\Omega_h)$ and $\mathcal{A}_A(\Omega_a)$ appropriate function spaces. The notation $\mathcal{I}_i(\boldsymbol{h})$ denotes the net current I_i flowing in (a group) of conductor(s) i ($i \in 1, \ldots, N = C$) for a given function $\boldsymbol{h}$, V_i is the associated voltage.

The choice of function spaces for the discretization will affect the stability of the method. In particular, functions with a non-zero trace on Γ_m should be chosen with care. Different possibilities are discussed in section III.

B. Thin conducting domains - t-a-formulation

The second formulation we consider is the so-called t-a-formulation for modelling thin superconducting tapes [2]. The tape is modelled as a line in 2D (a surface in 3D), denoted as Γ_w, and the current density inside the tape is described via a current vector potential whereas the external magnetic field, in a domain Ω_a, is expressed as the curl of a magnetic vector potential. The formulation writes:

From an initial solution, find $\boldsymbol{a} \in \mathcal{A}_A(\Omega_a)$ and $\boldsymbol{t} \in \mathcal{T}_I(\Gamma_w)$, s.t., for all time instant, and $\forall \boldsymbol{a}' \in \mathcal{A}_0(\Omega_a)$, $\forall \boldsymbol{t}' \in \mathcal{T}_0(\Gamma_w)$,

$$\begin{aligned} \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{a}\,,\mathbf{curl}\,\boldsymbol{a}'\right)_{\Omega_a} - \left(w\,\mathbf{curl}\,\boldsymbol{t}\,,\boldsymbol{a}'\right)_{\Gamma_w} &= 0, \\ \left(w\,\partial_t \boldsymbol{a}\,,\mathbf{curl}\,\boldsymbol{t}'\right)_{\Gamma_w} + \left(w\,\rho\,\mathbf{curl}\,\boldsymbol{t}\,,\mathbf{curl}\,\boldsymbol{t}'\right)_{\Gamma_w} & \\ = -\sum_{i\in C} V_i \mathcal{I}_i(\boldsymbol{t}'), & \end{aligned} \tag{2}$$

with $\mathcal{A}_A(\Omega_a)$ and $\mathcal{T}_I(\Gamma_w)$ appropriate function spaces, $\mathcal{I}_i(\boldsymbol{t}) = I_i$, the net current flowing in tape i for the potential $\boldsymbol{t}$, V_i the associated voltage, and w the tape width.

As with the coupled h-a-formulation, the choice of basis functions for discretization will affect the stability of the method. In particular, functions with a non-zero trace on Γ_w should be chosen with care.

III. Stability analysis

Both formulations take the form of a perturbed saddle-point problem. In addition to continuity and coerciveness conditions, such a problem is numerically stable when an inf-sup condition on the coupling operator is satisfied. For the h-a-formulation, the inf-sup condition is satisfied if there exists a β independent of mesh size such that

$$\inf_{\boldsymbol{a}\in H^{\perp}} \sup_{\boldsymbol{h}\in\mathcal{H}_{H,I}(\Omega_h)} \frac{\left(\boldsymbol{a}\times\boldsymbol{n}_{\Omega_h}\,,\boldsymbol{h}\right)_{\Gamma_\mathrm{m}}}{\|\boldsymbol{a}\|_{\mathcal{A}_A}\|\boldsymbol{h}\|_{\mathcal{H}_{H,I}}} \geq \beta > 0, \tag{3}$$

with $H^{\perp}$ the orthogonal complement of the kernel $H = \{a \in \mathcal{A}_A(\Omega_a) : (\boldsymbol{a}\times\boldsymbol{n}_{\Omega_h}\,,\boldsymbol{h})_{\Gamma_\mathrm{m}} = 0, \forall \boldsymbol{h} \in \mathcal{H}_{H,I}(\Omega_h)\}$. If $\beta \to 0$ for progressively refined meshes, stability issues might arise, such as spurious oscillations in the numerical solution. In this work, the inf-sup condition is evaluated at the discrete level with the numerical inf-sup test [3].

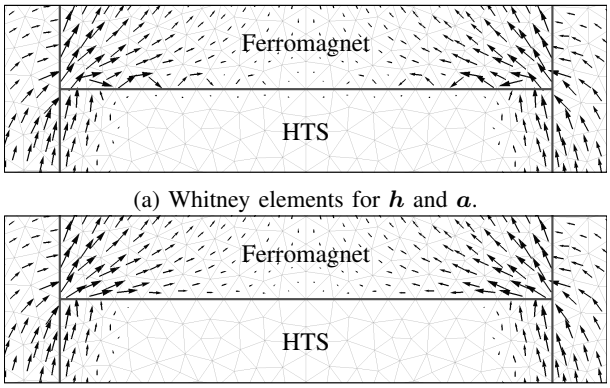
For naive choices of function spaces $\mathcal{H}_{H,I}(\Omega_h)$ and $\mathcal{A}_A(\Omega_a)$, we indeed observe such stability issues. An example is illustrated in Fig. 1a, for a two-bar problem. If $\boldsymbol{h}$ and $\boldsymbol{a}$ only use ("first-order") Whitney elements, spurious oscillations take place at the material interface. This is associated with an inf-sup constant decreasing to zero as the mesh is refined, as shown in Fig. 2.

One possibility to stabilize the problem is to enrich locally the function space of one of the two fields, for the h-a-formulation. This is illustrated in Fig. 1b, where using ("second-order") hierarchical basis functions for $\boldsymbol{h}$ on Γ_m allows to get rid of non-physical oscillations. Stability is indeed confirmed by the inf-sup test of Fig. 2.

Similar observations are made with the t-a-formulation: spurious oscillations in the current density are observed when using Whitney elements for $\boldsymbol{a}$ and $\boldsymbol{t}$ [4]. In that case, enriching the function space for $\boldsymbol{a}$ locally on Γ_w stabilizes the problem and numerical inf-sup tests confirm these results.

IV. Conclusion

In this work, we analyze the numerical stability of two mixed finite element formulations for systems with HTS: the h-a-formulation and the t-a-formulation. In both cases, the function spaces should be chosen carefully in order to avoid spurious oscillations in the numerical solution. We illustrated the numerical issues and the stabilized solution for the h-a-formulation. In the full paper, we will describe the analysis for the t-a-formulation as well. We will also define the function spaces in details.



(a) Whitney elements for $\boldsymbol{h}$ and $\boldsymbol{a}$.

(b) Whitney elements for $\boldsymbol{a}$. Enriched space for $\boldsymbol{h}$.

Fig. 1: Detail of a numerical solution. Magnetic flux density near the material interface (arrows represent the average value in each element) for a 2D two-bar problem (upper bar is a ferromagnet, lower bar is an HTS, exterior is air). (a) Unstable choice of function spaces, resulting in non-physical oscillations on Γ_m. (b) Example of a stabilized problem with hierarchical basis functions on Γ_m for $\boldsymbol{h}$.

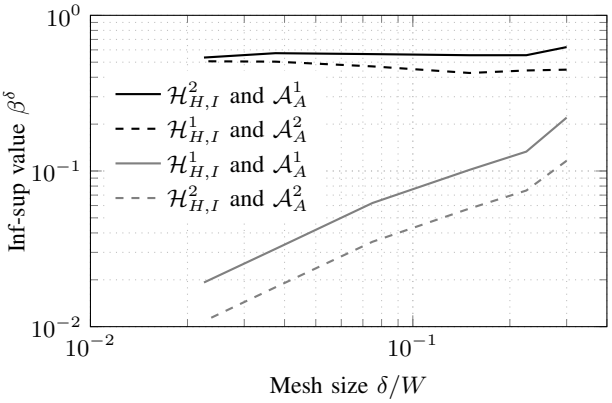


Fig. 2: Evolution of the inf-sup constant β^δ with mesh refinement ($\delta \to 0$) on the stacked bar linear problem. Four cases are considered. Superscript $\cdot^1$ refers to Whitney elements only and superscript $\cdot^2$ refers to enriched spaces. We can only conclude on stability when exactly one space is enriched with respect to Whitney elements.

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