3D stack magnetization problems: Solution by the FFT-based method

Leonid Prigozhin[1] & Vladimir Sokolovsky[2]

[1]Blaustein Institutes for Desert Research and [2]Physics Department

Ben-Gurion University of the Negev

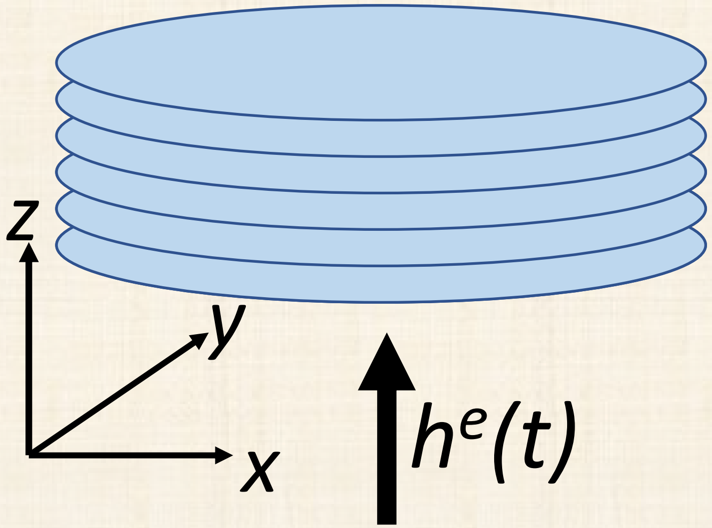
7-th Int. Workshop on Modelling in HTS, Nancy, June 22-23, 2021

FFT-based method for superconductivity problems:

- Derived for thin film problems (Vestgården et al. 2012, 2013):
  the method of lines in time with FFT-based discretization in space.
- Improved and extended to 3D bulk problems (LP and Sokolovsky, 2018).

This talk:

- Extension to stacks of thin flat films of the same (arbitrary) shape;
- Dense stacks of many films: homogenization;
- Simple formulation for high (infinite) stacks.



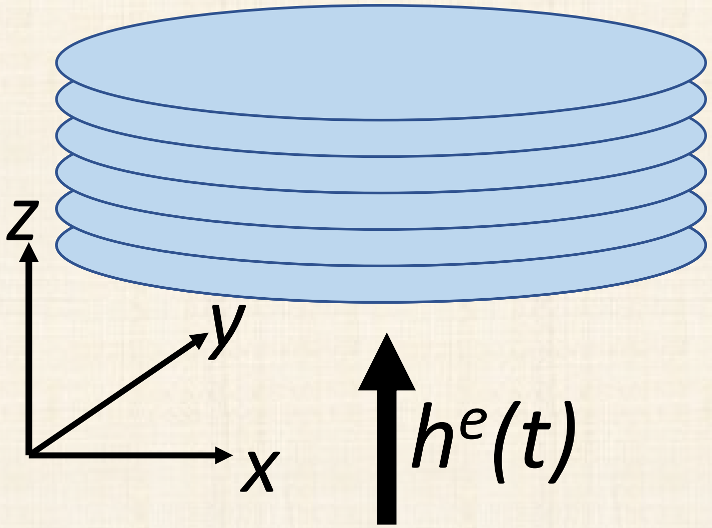
Problem formulation : A stack of thin sc films

$$\{(x, y, z_m) | (x, y) \in \Omega \subset R^2, \, z_m = md\}, \, m = 1, \ldots, N,$$

is in the uniform external field $\boldsymbol{h}^e = (0, 0, h_z^e(t))$;

we assume $\boldsymbol{e}_m = \rho\left(\left|\boldsymbol{j}_m\right|\right) \boldsymbol{j}_m$ for all films.

$$\nabla \cdot \boldsymbol{j}_m = 0 \text{ in } \Omega, \quad \boldsymbol{j}_m \cdot \boldsymbol{n}\,|_\Gamma = 0 \;\Rightarrow\; \boldsymbol{j}_m = \bar{\nabla} \times g_m, \quad g_m\,|_\Gamma = 0.$$



Problem formulation : A stack of thin sc films

$$\{(x,y,z_m)|(x,y)\in\Omega\subset R^2,\ z_m=md\},\ m=1,\dots,N,$$

is in the uniform external field $\boldsymbol{h}^e=(0,0,h_z^e(t))$;

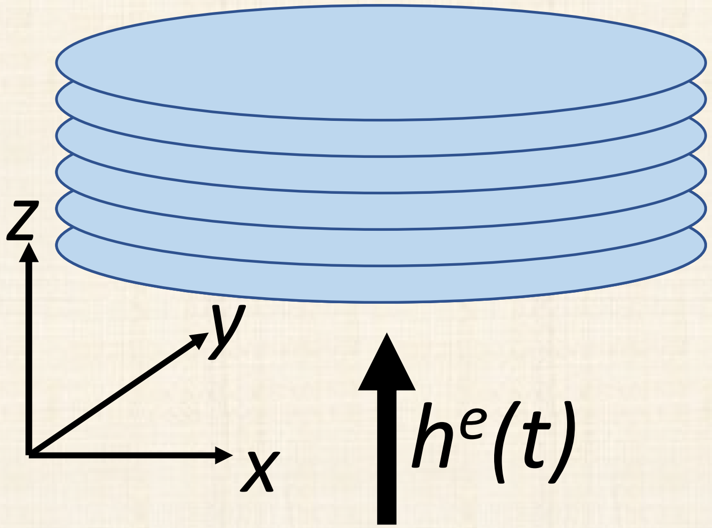
we assume $\boldsymbol{e}_m=\rho\left(|\boldsymbol{j}_m|\right)\boldsymbol{j}_m$ for all films.

$$\nabla\cdot\boldsymbol{j}_m=0 \text{ in } \Omega,\quad \boldsymbol{j}_m\cdot\boldsymbol{n}|_\Gamma=0\ \Rightarrow\ \boldsymbol{j}_m=\bar{\nabla}\times g_m,\quad g_m|_\Gamma=0.$$

Denoting $G_l(\boldsymbol{r})=\left(4\pi\sqrt{r^2+(ld)^2}\right)^{-1}$ we can write the Biot-Savart law as

$$h_{m,z}-h_z^e=\sum_{l=1}^{N}\nabla\times\int_\Omega G_{m-l}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{j}_l(\boldsymbol{r}',t)d\boldsymbol{r}'=-\sum_{l=1}^{N}\left(\partial_x G_{m-l}*\partial_x g_l+\partial_y G_{m-l}*\partial_y g_l\right),$$

where * means convolution and g_l are extended by zero to R^2.



Problem formulation : A stack of thin sc films

$\{(x,y,z_m)|(x,y)\in\Omega\subset R^2,\, z_m=md\},\ m=1,\ldots,N,$

is in the uniform external field $\boldsymbol{h}^e=(0,0,h_z^e(t))$;

we assume $\boldsymbol{e}_m=\rho\left(|\boldsymbol{j}_m|\right)\boldsymbol{j}_m$ for all films.

$\nabla\cdot\boldsymbol{j}_m=0$ in Ω, $\ \boldsymbol{j}_m\cdot\boldsymbol{n}|_\Gamma=0\ \Rightarrow\ \boldsymbol{j}_m=\bar{\nabla}\times g_m,\ \ g_m|_\Gamma=0.$

Denoting $G_l(\boldsymbol{r})=\left(4\pi\sqrt{r^2+(ld)^2}\right)^{-1}$ we can write the Biot-Savart law as

$$h_{m,z}-h_z^e=\sum_{l=1}^{N}\nabla\times\int_\Omega G_{m-l}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{j}_l(\boldsymbol{r}',t)d\boldsymbol{r}'=-\sum_{l=1}^{N}\left(\partial_x G_{m-l}*\partial_x g_l+\partial_y G_{m-l}*\partial_y g_l\right),$$

where * means convolution and g_l are extended by zero to R^2.

Fourier transform: $F[h_{m,z}-h_z^e](\boldsymbol{k})=|\boldsymbol{k}|^2\sum_{l=1}^{N}F[G_{m-l}](\boldsymbol{k})F[g_l](\boldsymbol{k}),\quad \boldsymbol{k}=(k_x,k_y).$

Substituting $F[G_{m-l}] = (2|k|)^{-1}\exp(-d\,|\boldsymbol{k}\,(m-l)|)$ into equation

$$F[h_{m,z} - h_z^e](\boldsymbol{k}) = |\boldsymbol{k}|^2 \sum_{l=1}^{N} F[G_{m-l}](\boldsymbol{k}) F[g_l](\boldsymbol{k})$$

we showed that for $\boldsymbol{k} \neq \boldsymbol{0}$ this system can be inverted and one can set

$$\boldsymbol{g} = F^{-1}\left[\frac{2}{|\boldsymbol{k}|}\boldsymbol{M}(\boldsymbol{k}) F\left[\boldsymbol{H}\right]\right] - \boldsymbol{C}(t),$$

where $\boldsymbol{g} = (g_1, \ldots, g_N)^T$, $\boldsymbol{H} = (h_{1,z} - h_z^e, \ldots, h_{N,z} - h_z^e)^T$,

$\boldsymbol{M}(\boldsymbol{k})$ is a three-diagonal matrix, $(2/|\boldsymbol{k}|)\boldsymbol{M}$ should be replaced by zero for $\boldsymbol{k} = \boldsymbol{0}$,

and time-dependent constants $C_m(t)$ are determined implicitly by the conditions

$$\int_{\Omega_{\text{out}}} g_m \mathrm{d}\boldsymbol{r} = 0$$

with $\Omega_{\text{out}} = R^2 \setminus \Omega$.

Differentiation with respect to time yields

$$\dot{\boldsymbol{g}} = F^{-1}\left[\frac{2}{|\boldsymbol{k}|}\boldsymbol{M}(\boldsymbol{k})F\left[\dot{\boldsymbol{H}}\right]\right] - \dot{\boldsymbol{C}}(t), \quad \int_{\Omega_{\text{out}}} \dot{\boldsymbol{g}}\,\mathrm{d}\boldsymbol{r} = \boldsymbol{0}. \tag{1}$$

Let the functions $g_m(t, \boldsymbol{r})$ be known at time t.

Differentiation with respect to time yields

$$\dot{\boldsymbol{g}} = F^{-1}\left[\frac{2}{|\boldsymbol{k}|}\boldsymbol{M}(\boldsymbol{k})F\left[\dot{\boldsymbol{H}}\right]\right] - \dot{\boldsymbol{C}}(t), \quad \int_{\Omega_{\text{out}}} \dot{\boldsymbol{g}}\,\mathrm{d}\boldsymbol{r} = \boldsymbol{0}. \qquad (1)$$

Let the functions $g_m(t, \boldsymbol{r})$ be known at time t.

By the Faraday law $\dot{h}_{m,z} = -\mu_0^{-1}\nabla \times \boldsymbol{e}_m$, so in films $\dot{h}_{m,z} = \mu_0^{-1}\nabla \cdot \left[\rho(|\nabla g_m|)\nabla g_m\right]$ and $\dot{H}_m = \dot{h}_{m,z} - \dot{h}_z^e$ is known in Ω. However, for (1) we need $\dot{H}_m$ in all R^2.

It is also necessary to ensure that all g_m remain zero outside Ω.

Differentiation with respect to time yields

$$\dot{\boldsymbol{g}} = F^{-1}\left[\frac{2}{|\boldsymbol{k}|}\boldsymbol{M}(\boldsymbol{k})F\left[\dot{\boldsymbol{H}}\right]\right] - \dot{\boldsymbol{C}}(t), \quad \int_{\Omega_{\text{out}}} \dot{\boldsymbol{g}}\,\mathrm{d}\boldsymbol{r} = \boldsymbol{0}. \qquad (1)$$

Let the functions $g_m(t, \boldsymbol{r})$ be known at time t.

By the Faraday law $\dot{h}_{m,z} = -\mu_0^{-1}\nabla \times \boldsymbol{e}_m$, so in films $\dot{h}_{m,z} = \mu_0^{-1}\nabla \cdot \left[\rho(|\nabla g_m|)\nabla g_m\right]$

and $\dot{H}_m = \dot{h}_{m,z} - \dot{h}_z^e$ is known in Ω. However, for (1) we need $\dot{H}_m$ in all R^2.

It is also necessary to ensure that all g_m remain zero outside Ω.

The problem to be solved is :

find $\dot{\boldsymbol{H}}$ in Ω_{out} such that (1) yields $\dot{\boldsymbol{g}} = \boldsymbol{0}$ in Ω_{out}.

This problem can be solved iteratively; its solution determines $\dot{g}$ in Ω.

The evolutionary problem for $\boldsymbol{g}$ is therefore defined.

Numerical implementation (Matlab) :

- We use a regular $N_x \times N_y$ grid in a rectangular domain containing Ω and several times larger; values of all variables are sought in the grid nodes.
- The continuous Fourier transform is replaced by the discrete one and computed using the FFT algorithm.
- All spacial derivatives are computed in the Fourier space; Gaussian smoothing is applied to supress high-frequency oscillations.
- A standard ODE solver is employed for integration in time.

Example.

In applications, superconducting film stacks often contain hundreds of densely packed films.

Homogenized anisotropic bulk model [Clem *et al* (2007)] was used by Kapolka *et al* (2018) and Olm *et al* (2019) to simulate magnetization of a dense $\Omega \times Nd = 10\times 10\times 1$ mm^3 stack of films, characterized by the power relation $e_m \sim (j_m / j_c)^{25}$ (the benchmark problem).

The bulk model assumes $e_{\|} \sim (J_{\|} / J_c)^{25}$ with $J_c = j_c / d$ in the parallel-to-films planes and an infinite resistivity in the normal direction.

High resistivity in z direction can slow down numerical simulations. An alternative [LP and Sokolovsky, 2011] is to consider a stack of $N_0 \ll N$ films with the distance $d_0 = Nd / N_0$ and the sheet critical current density, $j_{c0} = \left(j_c / d \right) d_0$.

For stacks of long strips an accurate approximation of the bulk model solution was observed if $d_0 / a_\Omega \leq 0.025$, where a_Ω is the strip width.

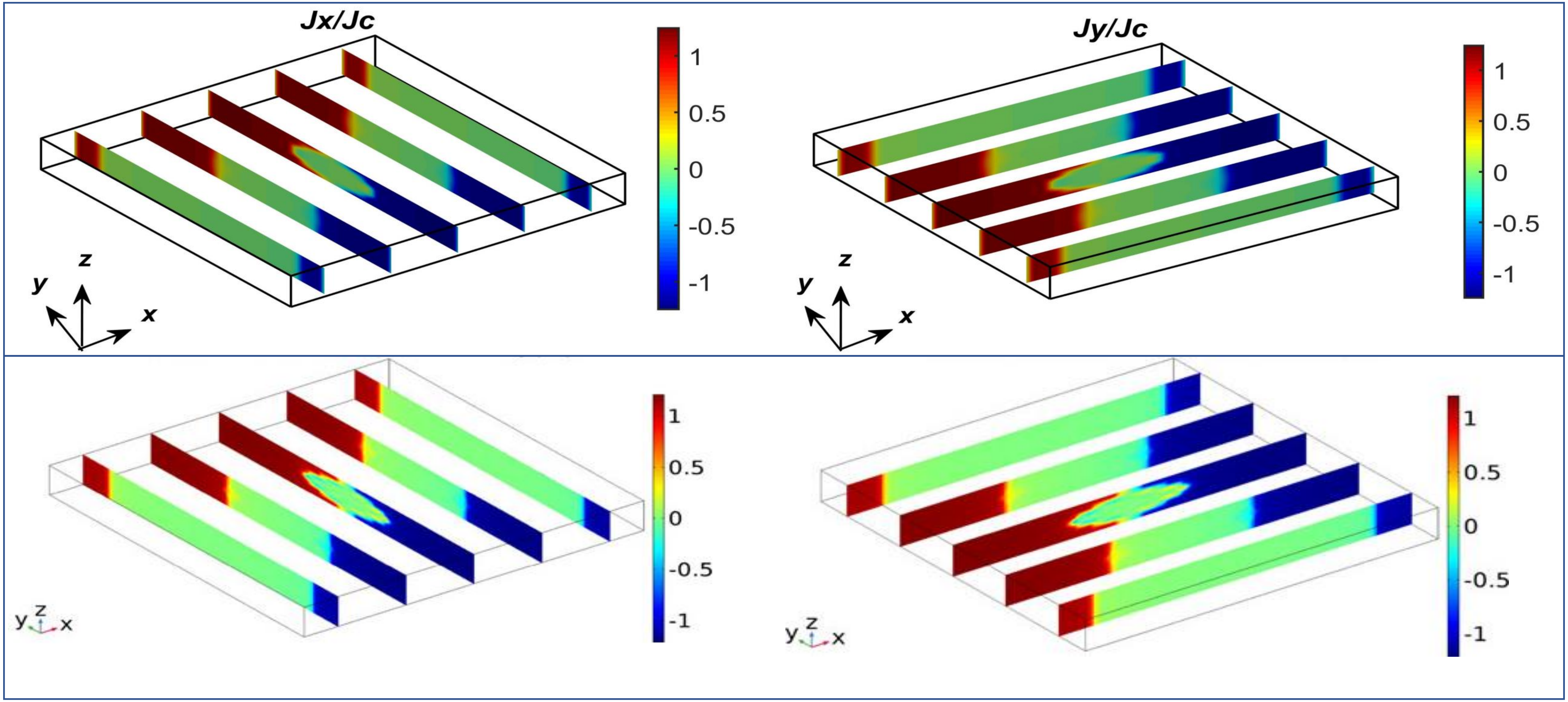
For the $10\times10\times1\ \mathrm{mm}^3$ stack, a 4-film stack already satisfies this criterion. We compared our results for losses, current densities, and computation times with those in Kapolka *et al* (2018) and Olm *et al* (2019) for the same sinusoidal external field, similar meshes inside the sc area, and similar PCs.

Two approaches to homogenization: reduced stack and anisotropic bulk (benchmark: a dense 10X10X1 mm^3 stack)

		Loss/cycle, Q_m (mJ)	**Computation time**
Stack of N films (our results)	N=4 N=6	3.43 3.51	9 hours 23 hours
Kapolka et al., 3 methods	MEMEP H-formulation, COMSOL VIEM	3.50 3.45 3.46	6 days* 1.7 days unknown
Olm et al.	H-formulation	3.46	unknown

*1 day on a computer cluster.

Computed losses per cycle in all cases are 3.46 mJ $\pm 1.5\%$.



Computed current density distributions at the peak of external field.

Top: Space-averaged solution for six-film stack (our result).

Bottom: Anisotropic bulk model solution (Kapolka *et al*, 2018).

High stacks

Previous 2D simulations showed that if the stack is high, the current densities are almost the same in all films except those close to stack top or bottom.

For the infinite stacks of long strips the problem was solved analytically by Mawatari (1996) for the Bean critical-state model.

For infinite stacks of arbitrary shaped films and any current-voltage relation, the FFT-based numerical method turns out to be almost the same as for a single film.

Infinite stacks. Since the currents are the same in all films,

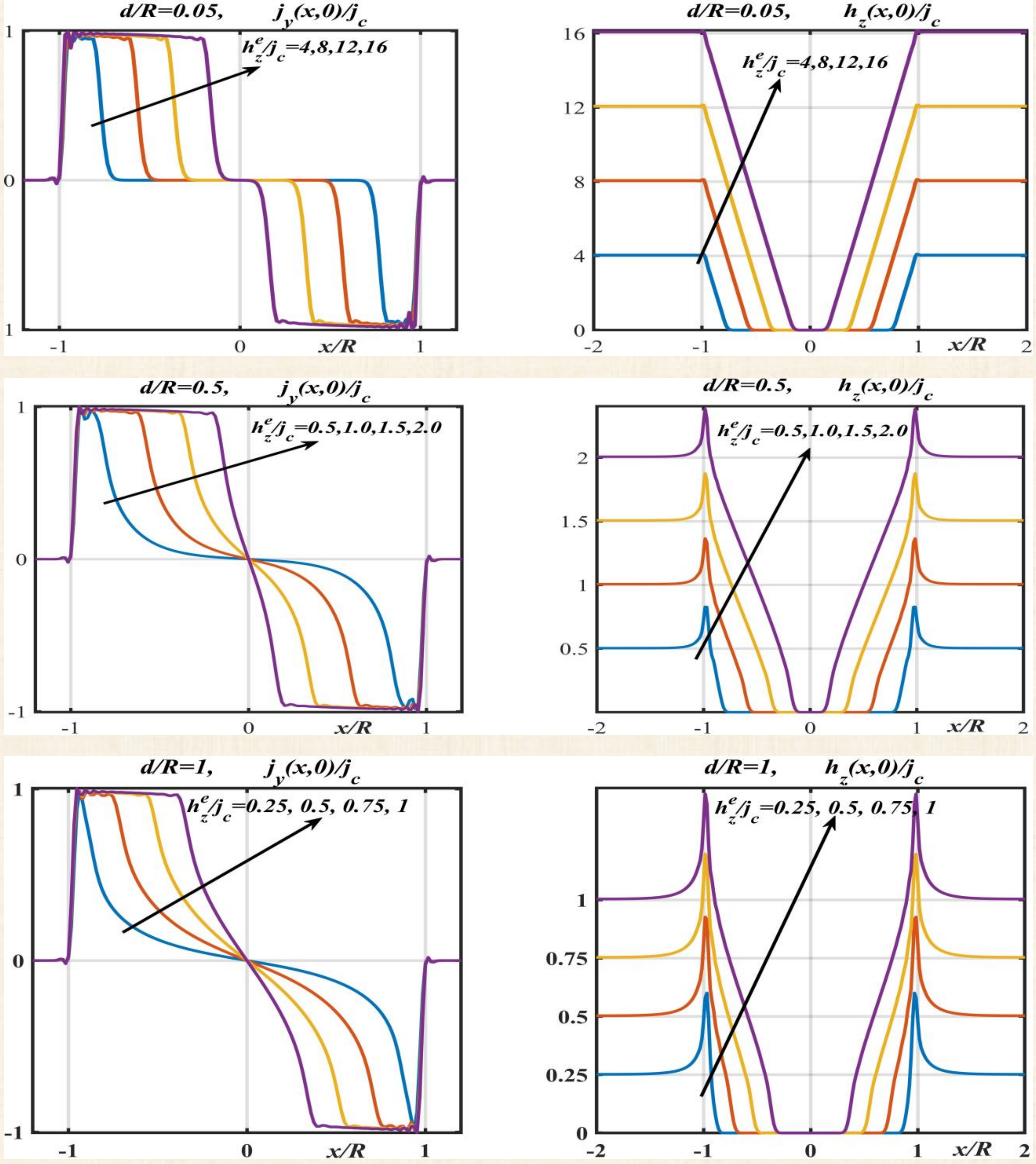
$$g_m = g \text{ for all } -\infty < m < \infty$$

the formulation simplifies. Summing up the influence of all films we obtained

$$\dot{g} = F^{-1}\left[S(|\boldsymbol{k}|)F\left[\dot{h}_z - \dot{h}_z^e\right]\right] - \dot{C}, \quad \int_{\Omega_{\text{out}}} \dot{g}\,\mathrm{d}\boldsymbol{r} = 0,$$

with $S(k) = \dfrac{2\left[1-\exp(-kd)\right]}{k\left[1+\exp(-kd)\right]}$.

This is very similar to the single film case, where $S(k) = \dfrac{2}{k}$, so numerical solution is similar too (LP and Sokolovsky, 2018).



For $d / R = 0.05$ (top row) solution is close to that for an infinite sc cylinder in a parallel field.

Example: $e \sim (j / j_c)^{50}$

Infinite stack of thin disks of radius *R* in a growing external field.

For $d / R = 1$ (bottom row) solution is close to that for a single sc disk.

Remarks on the FFT - based method

- The FFT-based method was extended to stacks of flat films of an arbitrary shape. We assumed a field-independent current-voltage relation but the method is not limited to such relations.

  For an infinite stack numerical solution by the FFT-based method is similar to that for a single film.

  Previously, efficiency of the FFT-based method was shown also for 3D bulk problems (and used for modeling magnetic lenses and magnetic shielding).

- Replacing a dense stack by a stack of only several films and rescaling (partial homogenization) can be more efficient than using the anisotropic bulk model.

Remarks on the FFT - based method

- The FFT-based method was extended to stacks of flat films of an arbitrary shape. We assumed a field-independent current-voltage relation but the method is not limited to such relations.

  For an infinite stack numerical solution by the FFT-based method is similar to that for a single film.

  Previously, efficiency of the FFT-based method was shown also for 3D bulk problems (and used for modeling magnetic lenses and magnetic shielding).

- Replacing a dense stack by a stack of only a few films and rescaling (partial homogenization + FFT method) can be more efficient than using the anisotropic bulk model.

Thank you!

Prigozhin and Sokolovsky, SuST, 31 (2018) 125001