

Stability of H-A and T-A coupled formulations

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June 23, 2021



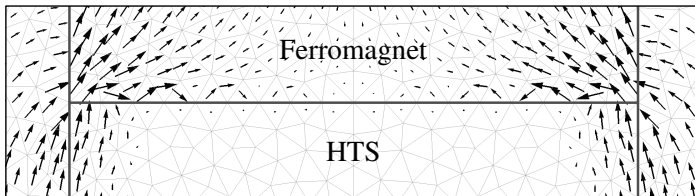
Introduction

Coupled formulations such as H-A or T-A often offer many advantages for HTS modeling:

- ▶ improved efficiency for nonlinear system resolution,
- ▶ reduced number of DOFs,
- ▶ increased flexibility,
- ▶ easier geometry definition. . .

However, they enter the framework of **mixed formulations**, thus requiring to be extremely careful regarding function spaces.

Otherwise, non-physical results must be expected:



Strong form

- ▶ Magnetodynamic (quasistatic) equations

$$\operatorname{div} \mathbf{b} = 0, \quad \operatorname{curl} \mathbf{h} = \mathbf{j}, \quad \operatorname{curl} \mathbf{e} = -\partial_t \mathbf{b}.$$

- ▶ Constitutive relationships

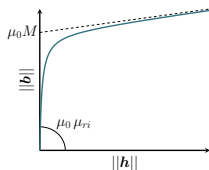
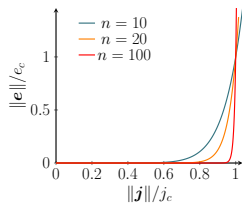
High-temperature superconductors (HTS):

$$\mathbf{e} = \rho(\|\mathbf{j}\|)\mathbf{j} \quad \text{and} \quad \mathbf{b} = \mu_0 \mathbf{h},$$

with the power law $\rho(\|\mathbf{j}\|) = \frac{e_c}{j_c} \left(\frac{\|\mathbf{j}\|}{j_c} \right)^{n-1}$.

Ferromagnetic material (FM):

$$\mathbf{b} = \mu(\mathbf{b}) \mathbf{h} \quad \text{and} \quad \mathbf{j} = \mathbf{0}.$$



Dual formulations

Two classes of formulations with the **finite element method**:

- ▶ h -conform, e.g. **h -formulation**,
 - ▶ enforces the continuity of the tangential component of \mathbf{h} ,
 - ▶ involves $\mathbf{e} = \rho \mathbf{j}$ and $\mathbf{b} = \mu \mathbf{h}$,
 - ▶ with $\mathbf{curl} \mathbf{h} = \mathbf{0}$ in non-conducting domain (" \mathbf{h} - ϕ " + cuts),

$$(\partial_t(\mu \mathbf{h}), \mathbf{h}')_{\Omega} + (\rho \mathbf{curl} \mathbf{h}, \mathbf{curl} \mathbf{h}')_{\Omega_c} - \langle \mathbf{e} \times \mathbf{n}, \mathbf{h}' \rangle_{\Gamma_e} = 0.$$

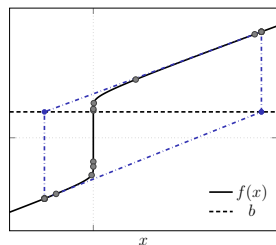
- ▶ b -conform, e.g. **a -formulation**,
 - ▶ enforces the continuity of the normal component of \mathbf{b} ,
 - ▶ involves $\mathbf{j} = \sigma \mathbf{e}$ and $\mathbf{h} = \nu \mathbf{b}$, ($\sigma = \rho^{-1}$, $\nu = \mu^{-1}$)

$$(\nu \mathbf{curl} \mathbf{a}, \mathbf{curl} \mathbf{a}')_{\Omega} + (\sigma \partial_t \mathbf{a}, \mathbf{a}')_{\Omega_c} - \langle \mathbf{h} \times \mathbf{n}_{\Omega}, \mathbf{a}' \rangle_{\Gamma_h} = 0.$$

Nonlinear constitutive laws involved in **opposite ways** \Rightarrow **very different numerical behaviors** are expected... and observed.

Best choice for HTS only

Cycles in iterations:



In the **a -formulation**, the diverging slope associated with $j = \sigma e$ for $e \rightarrow 0$ is really difficult to handle.

⇒ Among the two simple formulations, the **h -formulation** is much more efficient for systems with HTS:

- ▶ with an **adaptive time-stepping** algorithm,
- ▶ solved with a **Newton-Raphson** method.

Dular, J., et al. (2020) TAS 30 8200113.

Ferromagnetic materials

The nonlinearity is in the magnetic constitutive law.

- ▶ ***h*-formulation** the involved law is $\mathbf{b} = \mu \mathbf{h}$.



⇒ Often enters **cycles** with Newton-Raphson.
OK with fixed point, or N-R with relaxation factors but slow.

- ▶ ***a*-formulation** the involved law is $\mathbf{h} = \nu \mathbf{b}$.



⇒ Efficiently solved with Newton-Raphson.

The ***a*-formulation** is more appropriate for dealing with the nonlinearity, whereas for HTS, the ***h*-formulation** is best.

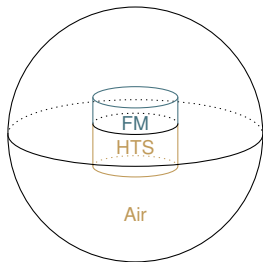
Coupled materials - h - a -formulation

Use the best formulation in each material

Decompose the domain Ω , for example into:

- ▶ $\Omega^h = \{\text{HTS, Air}\}$
- ▶ $\Omega^a = \{\text{Ferromagnet}\}$

and couple via $\Gamma_m = \partial(\text{FM})$:



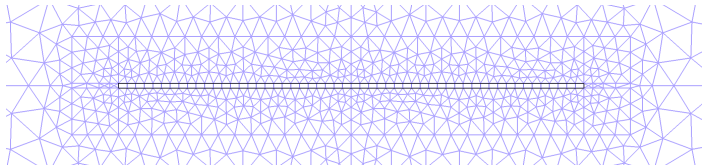
$$\begin{aligned}(\partial_t(\mu \mathbf{h}), \mathbf{h}')_{\Omega^h} + (\rho \mathbf{curl} \mathbf{h}, \mathbf{curl} \mathbf{h}')_{\Omega_c^h} + \langle \partial_t \mathbf{a} \times \mathbf{n}_{\Omega^h}, \mathbf{h}' \rangle_{\Gamma_m} &= 0, \\ \langle \mathbf{h} \times \mathbf{n}_{\Omega^a}, \mathbf{a}' \rangle_{\Gamma_m} - (\nu \mathbf{curl} \mathbf{a}, \mathbf{curl} \mathbf{a}')_{\Omega^a} &= 0.\end{aligned}$$

Dular, J., et al. (2020) TAS 30 8200113.
See also: Brambila R. et al, (2018) TAS 28, 5207511.

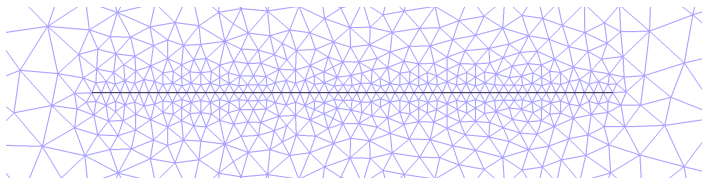
HTS tapes - t - a -formulation

To model thin superconducting tapes, two main possibilities:

1. Use the true geometry and the h -formulation with one-element across the thickness (quad., prism, hexa.).



2. Perform a thin shell approximation and model the tape as a line \Rightarrow t - a -formulation.

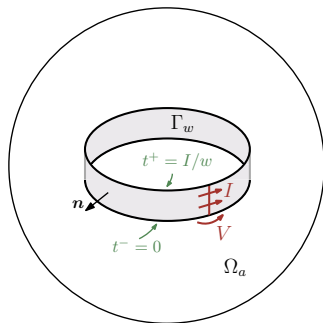


HTS tapes - t - a -formulation

Consider an HTS tape Γ_w of thickness w .

The current density is described by a **current potential** t :

- ▶ such that $\mathbf{j} = \mathbf{curl} \, t$,
- ▶ gauged by being defined along the normal of the tape, $\mathbf{t} = t \, \mathbf{n}$,
- ▶ with BC related to the total current I ($t^+ - t^- = I/w$).



$$\begin{aligned}(\nu \mathbf{curl} \, \mathbf{a} , \mathbf{curl} \, \mathbf{a}')_{\Omega_a} - \langle w \mathbf{curl} \, \mathbf{t} , \mathbf{a}' \rangle_{\Gamma_w} &= 0, \\ \langle w \partial_t \mathbf{a} , \mathbf{curl} \, \mathbf{t}' \rangle_{\Gamma_w} + \langle w \rho \mathbf{curl} \, \mathbf{t} , \mathbf{curl} \, \mathbf{t}' \rangle_{\Gamma_w} &= 0,\end{aligned}$$

Perturbed saddle point problems

h - a -formulation

$$\begin{aligned} \langle \partial_t(\mu \mathbf{h}), \mathbf{h}' \rangle_{\Omega^h} + (\rho \operatorname{curl} \mathbf{h}, \operatorname{curl} \mathbf{h}')_{\Omega_c^h} + \langle \partial_t \mathbf{a} \times \mathbf{n}_{\Omega^h}, \mathbf{h}' \rangle_{\Gamma_m} &= 0, \quad \forall \mathbf{h}' \in \mathcal{H}, \\ \langle \mathbf{h} \times \mathbf{n}_{\Omega^a}, \mathbf{a}' \rangle_{\Gamma_m} - (\nu \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega^a} &= 0, \quad \forall \mathbf{a}' \in \mathcal{A}. \end{aligned}$$

t - a -formulation

$$\begin{aligned} \langle w \rho \operatorname{curl} \mathbf{t}, \operatorname{curl} \mathbf{t}' \rangle_{\Gamma_w} + \langle w \partial_t \mathbf{a}, \operatorname{curl} \mathbf{t}' \rangle_{\Gamma_w} &= 0, \quad \forall \mathbf{t}' \in \mathcal{T}, \\ \langle w \operatorname{curl} \mathbf{t}, \mathbf{a}' \rangle_{\Gamma_w} - (\nu \operatorname{curl} \mathbf{a}, \operatorname{curl} \mathbf{a}')_{\Omega^a} &= 0, \quad \forall \mathbf{a}' \in \mathcal{A}. \end{aligned}$$

These are **mixed** formulations, **perturbed saddle point** problems:

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v \rangle, & \forall v \in V, \\ b(u, q) - c(p, q) = \langle g, q \rangle, & \forall q \in Q, \end{cases} \quad \text{or} \quad \begin{pmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}.$$

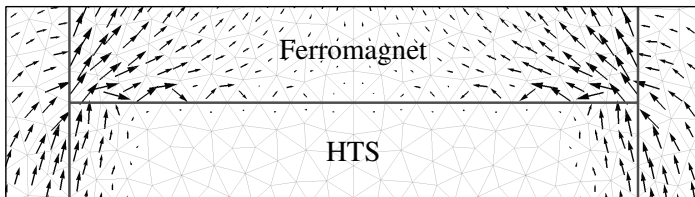
⇒ **Compatibility conditions** to ensure numerical stability.

D. Boffi, F. Brezzi, et al., Mixed FE methods and applications, Springer, 2013.

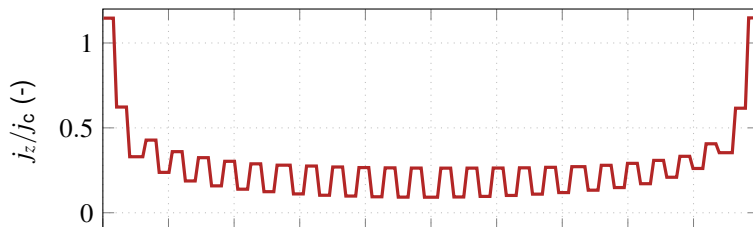
Numerical oscillations

If function spaces **do not satisfy** the compatibility conditions. . .

h - a -formulation First-order functions for h and a :



t - a -formulation First-order functions for t and a :



Compatibility conditions

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}.$$

The solution is **stable**, i.e., $\|\mathbf{u}\|_V + \|\mathbf{p}\|_Q \leq C(\|\mathbf{f}\|_{V'} + \|\mathbf{g}\|_{Q'})$, if $\exists \alpha, \beta, \gamma > 0$ (strictly) such that

$$\mathbf{v}^T \mathbf{A} \mathbf{v} \geq \alpha \|\mathbf{v}\|^2, \quad \forall \mathbf{v} \in \ker(\mathbf{B}),$$

$$\mathbf{q}^T \mathbf{C} \mathbf{q} \geq \gamma \|\mathbf{q}\|^2, \quad \forall \mathbf{q} \in \ker(\mathbf{B}^T),$$

$$\inf_{\mathbf{q} \in (\ker(\mathbf{B}^T))^\perp} \sup_{\mathbf{v} \in V} \frac{\mathbf{q}^T \mathbf{B} \mathbf{v}}{\|\mathbf{q}\|_Q \|\mathbf{v}\|_V} \geq \beta > 0 \quad (\text{inf-sup condition}).$$

In our case, the **inf-sup condition** is the most restrictive.

Inf-sup test

The inf-sup condition is not easy to check analytically.

⇒ We perform a **numerical inf-sup test**.

On progressively refined meshes, for given function spaces:

1. Define suitable norms.
2. Extract matrices \mathbf{B} , \mathbf{N}_V , and \mathbf{N}_Q , from the FE assembly, with

$$\|\mathbf{v}\|_V^2 = \mathbf{v}^T \mathbf{N}_V \mathbf{v},$$

$$\|\mathbf{q}\|_Q^2 = \mathbf{q}^T \mathbf{N}_Q \mathbf{q}.$$

3. Solve the eigenvalue problem

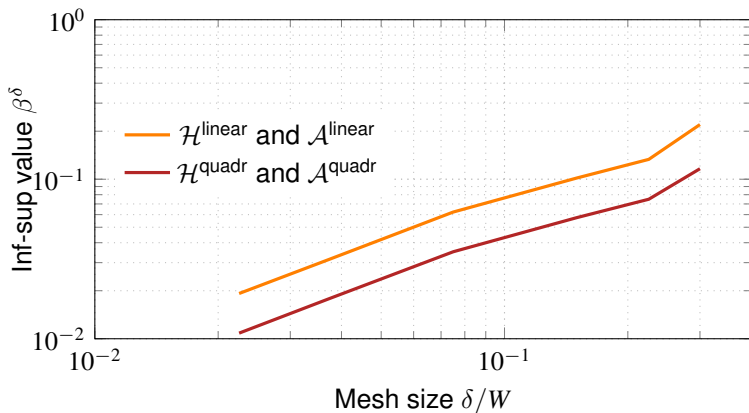
$$\left(\mathbf{B} \mathbf{N}_V^{-1} \mathbf{B}^T \right) \mathbf{q} = \lambda \mathbf{N}_Q \mathbf{q}.$$

How does the **lowest non-zero eigenvalue** behave when the mesh is refined?

- ▶ It tends to zero ⇒ **unstable**,
- ▶ It is bounded from below ⇒ **stable**.

h - a -formulation Unstable choices

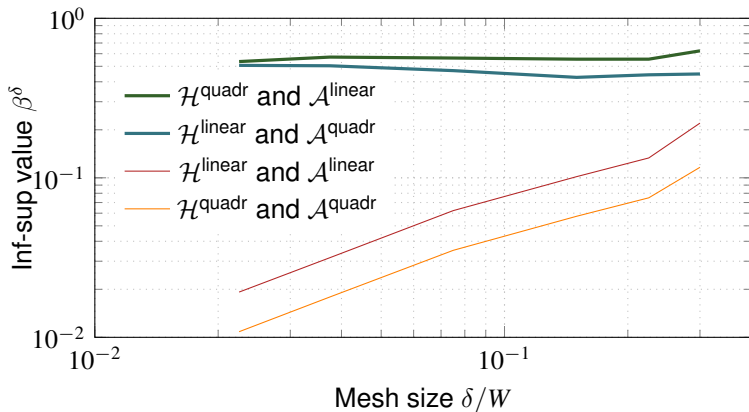
Linear or quadratic elements for both h and $a \Rightarrow$ Unstable.



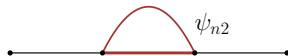
h - a -formulation Stable choices

One way to stabilize the problem:

⇒ Increase the discretization order of **one** field (**h** or **a**).

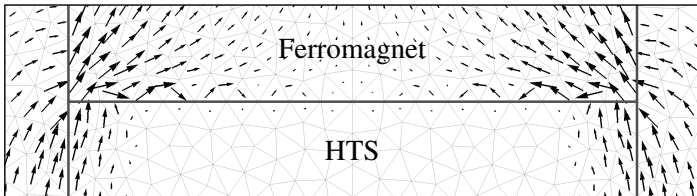


Increasing the order on the coupling interface only is sufficient.

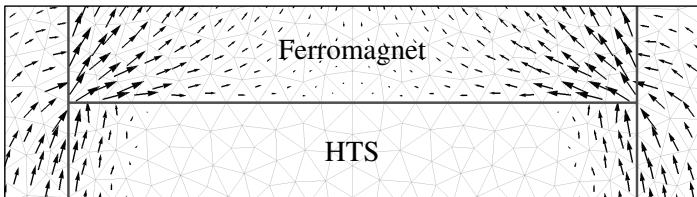


h - a -formulation Stabilization

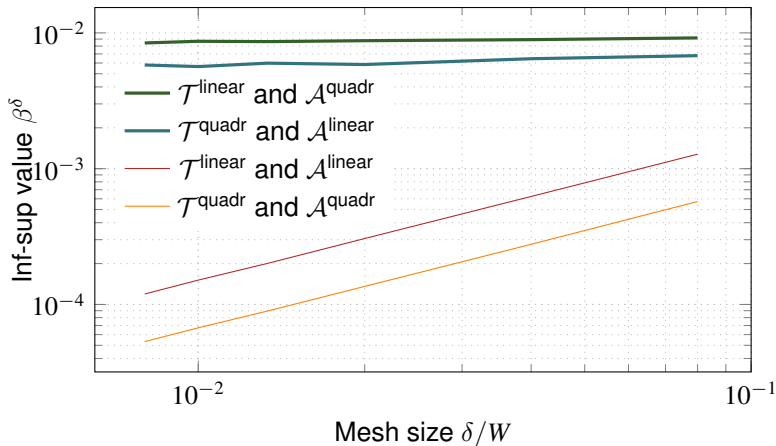
- ▶ First-order functions for h and a (inf-sup KO):



- ▶ Second-order for a , first-order for h (inf-sup OK):



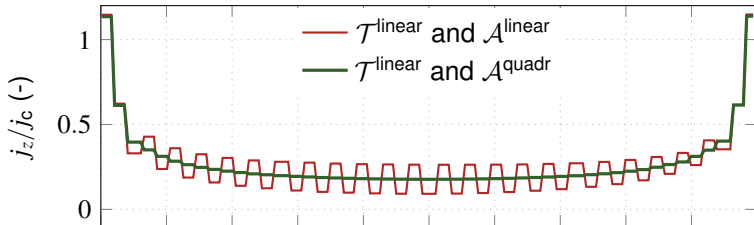
t - a -formulation - Inf-sup test



Similar conclusions: increase the order of **one** function space.

t - a -formulation - Stabilization

Example for a 2D case, current density along the tape:



NB: With the choice $\mathcal{T}^{\text{quadr}}$ and $\mathcal{A}^{\text{linear}}$, Newton-Raphson faces convergence difficulties.

Conclusion and application

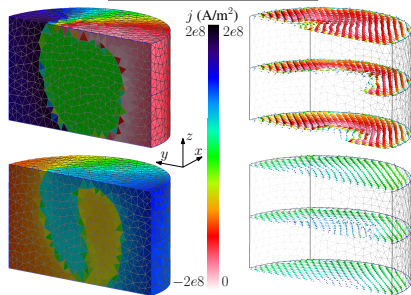
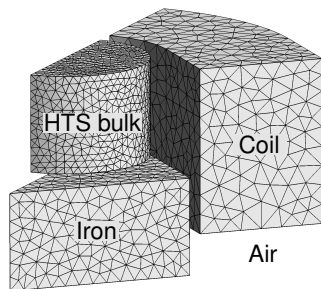
Two coupled formulations

- ▶ Modeling HTS and FM efficiently \Rightarrow h - a -formulation
- ▶ Modeling HTS tapes can be done with a t - a -formulation

Both formulations are **mixed** \Rightarrow Inf-sup condition.

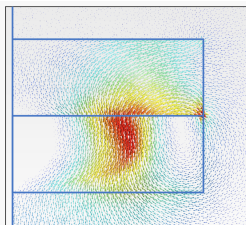
By enriching one of the two spaces, stability is guaranteed.

Extension to 3D: Bulk magnetization with h - a -formulation (h - ϕ)



References

- ▶ Life-HTS website: <http://www.life-hts.uliege.be/>
- ▶ *Mixed and hybrid finite element methods*,
F. Brezzi, M. Fortin, Springer Science & Business Media (2012).
- ▶ *On the Stability of Mixed Finite-Element Formulations for High-Temperature Superconductors*,
J. Dular, M. Harutyunyan, L. Bortot, S. Schöps, B. Vanderheyden, and C. Geuzaine (to be published).
- ▶ *A coupled A–H formulation for magneto-thermal transients in high-temperature superconducting magnets*,
L. Bortot, B. Auchmann, I. C. Garcia, H. De Gersem, M. Maciejewski, M. Mentink, S. Schöps, J. Van Nugteren, and A. Verweij, TAS 30 (5), 1-11.



Cylinders model after magnetization.

Life-HTS

Liège university Finite Element models for High-Temperature Superconductors

This project contains model files for modeling systems containing high-temperature superconductors (HTS) with [GetDP](#) as a finite element solver and [Gmsh](#) as mesh generator.

[Files are available here.](#)

Several finite element formulations are implemented together with various linearization methods and iterative procedures. Simple models are proposed for practical applications (bulk and tapes HTS, coupling with ferromagnets...)

These models are developed at the University of Liège.